Solitons of Sigma Model on Noncommutative Space as Solitons of Electron System

Hideharu Otsu *
Faculty of Economics, Aichi University,
Toyohashi, Aichi 441-8522, Japan

Toshiro Sato †
Faculty of Law and Economics, Mie Chukyo University,
Matsusaka, Mie 515-8511, Japan

Hitoshi Ikemori ‡
Faculty of Economics, Shiga University,
Hikone, Shiga 522-8522, Japan

Shinsaku Kitakado §
Department of Physics, Faculty of Science and Technology,
Meijo University,
Tempaku, Nagoya 486-8502, Japan

Abstract

We study the relationship of soliton solutions for electron system with those of the sigma model on the noncommutative space, working directly in the operator formalism. We find that some soliton solutions of the sigma model are also the solitons of the electron system and are classified by the same topological numbers.

*otsu@vega.aichi-u.ac.jp
†tsato@mie-chukyo-u.ac.jp
‡ikemori@biwako.shiga-u.ac.jp
§kitakado@ccmfs.meijo-u.ac.jp

1 Introduction

As is well known, the systems in the strong magnetic field can be described by the noncommutative field theories. This picture has an application to the solid state physics and effective theories derived from the string model also belong to this category. Various analyses of field theories on the noncommutative spaces are performed [1][2][3]. In particular, nonlinear sigma model on the noncommutative plane are investigated in detail and the structures of soliton solutions are becoming clear [4][5][6][7][8][9][10][11]. The soliton solutions that do not have the couterparts in the commutative space are also known to exist in this model [10][11].

In this paper, we shall investigate the physical phenomena related to the soliton solutions of the nonlinear sigma model. We consider two dimensional spinning electron system in the magnetic field perpendicular to the plane. Coulomb repulsion is considered to be present among the electrons. If we restrict our consideration to the lowest Landau level (LLL), this system is reduced to that on the noncommutative space [12][13][14]. In ref.[12], the Hamiltonian for the electron system has been constructed, where $O(3)\sigma$ model on the commutative space has been derived through the expansion of the $*$product and the application of the soliton solution was also considered. We shall study the soliton solution directly in the operator formalism. We find that some soliton solutions of the sigma model on the noncommutative space are also the solitons of the electron system. The configuration space of the electron system can be identified with that of $O(3)\sigma$ model when expressed in terms of the projector P, thus the same topological numbers can be used in classifying the configurations.

In section 2, we summarize the properties of the soliton solutions of nonlinear sigma model on the noncommutative plane. In section 3, we study the noncommutative BPS equations and their soliton solutions in the operator form considering the Hamiltonian for the electron system. In section 4, we discuss the classification of solitons by the new topological number. Finally, in section 5, we conclude with summary and discussions.

2 Nonlinear Sigma Model on Noncommutative Plane and Soliton Solutions

In this section we summarize the main properties of soliton solutions on the noncommutative two dimensional space. First we fix the notations. The space coordinates obey the commutation relation

$$[x, y] = i\theta \tag{1}$$

or

$$[z, \bar{z}] = \theta > 0, \tag{2}$$

when written in the complex variables, $z = \frac{1}{\sqrt{2}}(x+iy)$ and $\bar{z} = \frac{1}{\sqrt{2}}(x-iy)$. The Hilbert space can be described in terms of the energy eigenstates $|n\rangle$ of the harmonic oscillator whose creation and annihilation operators are $\bar{z}$ and z respectively,

$$\begin{aligned} z\,|n\rangle &= \sqrt{\theta n}\,|n-1\rangle\,, \\ \bar{z}\,|n\rangle &= \sqrt{\theta(n+1)}\,|n+1\rangle\,. \end{aligned} \tag{3}$$

Space integrals on the commutative space are replaced by the trace on the Hilbert space

$$\int d^2x \Rightarrow \mathrm{Tr}_{\mathcal{H}}, \tag{4}$$

where $\mathrm{Tr}_{\mathcal{H}}$ denotes the trace over the Hilbert space as

$$\mathrm{Tr}_{\mathcal{H}}\mathcal{O} = 2\pi\theta \sum_{n=0}^{\infty} \langle n|\,\mathcal{O}\,|n\rangle\,. \tag{5}$$

The derivatives with respect to z and $\bar{z}$ are defined by $\partial_z = -\theta^{-1}\,[\bar{z},]$ and $\partial_{\bar{z}} = \theta^{-1}\,[z,]$.

Next we turn to the nonlinear sigma model on the noncommutative space. As a field variable we take the 2×2 matrix projector P ($P^2 = P$). Lagrangian and topological charge are respectively

$$L = \frac{1}{2}\mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}(\partial_t P)^2\right] - \frac{1}{\theta^2}\mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}\left([z,P][P,\bar{z}]\right)\right] \tag{6}$$

and

$$Q = \frac{1}{4\pi\theta^2}\mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}\left\{(2P-1)\left([\bar{z},P][z,P]-[z,P][\bar{z},P]\right)\right\}\right]. \tag{7}$$

Energy for the static configuration is expressed as

$$E = \frac{1}{\theta^2}\mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}([z,P]\,[P,\bar{z}])\right], \tag{8}$$

which leads to the energy bound [6][15]

$$E \geq 2\pi\,|Q|\,. \tag{9}$$

For $Q>0$, the BPS soliton equation [6][15][16] is

$$(1-P)zP = 0. \tag{10}$$

Similarly, for $Q<0$ the BPS anti-soliton equation is

$$(1-P)\bar{z}P = 0. \tag{11}$$

The static configurations are classified by the topological charge $Q = 0, \pm 1, \pm 2, \cdots$ and $\langle \mathrm{tr} P \rangle_\infty = 0, 1, 2$, the values of $\mathrm{tr}P$ at the boundary of the Hilbert space [11], where $\langle \mathrm{tr} P \rangle_\infty$ is defined as

$$\langle \mathrm{tr} P \rangle_\infty \equiv \lim_{n\to\infty} \langle n| \, \mathrm{tr} P \, |n\rangle \, . \tag{12}$$

The following soliton solutions that satisfy the BPS equations are known. The solitons that extrapolate into those on the commutative plane with $Q = n, \ E = 2\pi n$ and $\langle \mathrm{tr} P \rangle_\infty = 1$ are [4]

$$P = \begin{pmatrix} z^n \dfrac{1}{\bar{z}^n z^n + 1} \bar{z}^n & z^n \dfrac{1}{\bar{z}^n z^n + 1} \\ \dfrac{1}{\bar{z}^n z^n + 1} \bar{z}^n & \dfrac{1}{\bar{z}^n z^n + 1} \end{pmatrix} . \tag{13}$$

The anti-solitons extrapolating into those on the commutative plane with $Q = -n, \ E = 2\pi n$ and $\langle \mathrm{tr} P \rangle_\infty = 1$ are

$$P = \begin{pmatrix} \bar{z}^n \dfrac{1}{z^n \bar{z}^n + 1} z^n & \bar{z}^n \dfrac{1}{z^n \bar{z}^n + 1} \\ \dfrac{1}{z^n \bar{z}^n + 1} z^n & \dfrac{1}{z^n \bar{z}^n + 1} \end{pmatrix} . \tag{14}$$

The commutative limit of which are obtained by simply substituting the c-number z and $\bar{z}$ into (13) and (14), then they reduce to the solitons and anti-solitons of the nonlinear sigma model respectively. Furthermore, there exist solitons that do not have the counterparts on the commutative space. They have $Q = n, \ E = 2\pi n$ and $\langle \mathrm{tr} P \rangle_\infty = 1$ for soliton solutions expressed as [10][11]

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \sum_{m=0}^{n-1} |m\rangle \langle m| \end{pmatrix} , \tag{15}$$

$Q = -n, \ E = 2\pi n$ and $\langle \mathrm{tr} P \rangle_\infty = 1$ for anti-soliton solutions;

$$P = \begin{pmatrix} 1 - \sum_{m=0}^{n-1} |m\rangle \langle m| & 0 \\ 0 & 0 \end{pmatrix} , \tag{16}$$

$Q = k + n, \ E = 2\pi(k + n)$ and $\langle \mathrm{tr} P \rangle_\infty = 0$ for soliton solutions;

$$P = \begin{pmatrix} \sum_{m=0}^{k-1} |m\rangle \langle m| & 0 \\ 0 & \sum_{m=0}^{n-1} |m\rangle \langle m| \end{pmatrix} \tag{17}$$

and $Q = -(k+n), \ E = 2\pi(k+n)$ and $\langle \mathrm{tr} P \rangle_\infty = 2$ for anti-soliton solutions;

$$P = \begin{pmatrix} 1 - \sum_{m=0}^{n-1} |m\rangle \langle m| & 0 \\ 0 & 1 - \sum_{m=0}^{k-1} |m\rangle \langle m| \end{pmatrix} . \tag{18}$$

For the finite energy configuration, the topological charge Q can be rewritten after some calculations into the following simple form

$$Q = \frac{1}{2\pi\theta} \mathrm{Tr}_{\mathcal{H}} \left(\mathrm{tr} P - \langle \mathrm{tr} P \rangle_\infty \right) . \tag{19}$$

In fact, we can easily confirm that the soliton solutions from (13) to (18) when substituted into (19) give the same value for Q calculated with (7).

3 Noncommutative View of Electron System on the Plane

In this section we consider a two dimensional system of electrons with spin in the magnetic field B perpendicular to the plane. Coulomb repulsive force among the electrons is assumed. If we restrict the electron states to the lowest Landau level, the system is reduced to that on the noncommutative plane [12][13][14].

Let us express the external magnetic field in terms of the vector potential

$$A_x = -\frac{By}{2}, \ A_y = \frac{Bx}{2}, \tag{20}$$

where x and y are the coordinates on the plane. We can define the independent oscillators in terms of the complex variables, $z = \frac{1}{\sqrt{2}}(x+iy)$ and $\bar{z} = \frac{1}{\sqrt{2}}(x-iy)$ as

$$\begin{aligned} a &= \theta\partial_{\bar{z}} + \frac{z}{2}, \ a^\dagger = -\theta\partial_z + \frac{\bar{z}}{2}, \\ b &= \theta\partial_z + \frac{\bar{z}}{2}, \ b^\dagger = -\theta\partial_{\bar{z}} + \frac{z}{2}, \end{aligned} \tag{21}$$

which satisfy

$$[a, a^\dagger] = [b, b^\dagger] = \theta, \tag{22}$$

where $\theta = 1/(eB)$ [12][13]. The Hamiltonian of the first quantized system is proportional to $Ba^\dagger a$. Thus, a and $a^\dagger$ induce the transitions among the Landau levels, while b and $b^\dagger$ induce the transitions within each Landau level.

As is well known, in the large B limit, the system can be restricted to the lowest Landau level (LLL), and the electrons in the LLL can be considered as a system on the plane with the noncommutative coordinates b and $b^\dagger$. The Hilbert space is spanned by $\{|n\rangle\}$, for which

$$b^\dagger b \, |n\rangle = n\theta \, |n\rangle \, . \tag{23}$$

The second quantized field that annihilates (creates) an electron with a spin σ at position $\vec{x}$ in the LLL can be constructed as

$$\Psi_\sigma\left(\vec{x}\right) = \sum_n \left\langle \vec{x} | n \right\rangle C_{n\sigma} \ , \ \Psi_\sigma^\dagger = \left(\Psi_\sigma\right)^\dagger \ , \tag{24}$$

in terms of the fermionic operators $C_{n\sigma}$ $\left(C_{n\sigma}^{\dagger}\right)$ which annihilates (creates) an electron in the n-th orbital, where

$$\langle\vec{x}|n\rangle=\left(\frac{z}{\sqrt{\theta}}\right)^{n}\frac{1}{\sqrt{2\pi n!}}\exp\left(-\frac{\bar{z}z}{2\theta}\right)\ . \tag{25}$$

These satisfy the anticommutation relation

$$\left\{\Psi_{\sigma}\left(\vec{x}\right),\Psi_{\sigma'}^{\dagger}\left(\vec{x}\right)\right\}=\rho\delta_{\sigma\sigma'}\ , \tag{26}$$

where $\rho=\langle\vec{x}|\vec{x}\rangle=(2\pi\theta)^{-1}$. As we shall see, the system can be described in terms of P-field that appeared in previous section which is expressed using the electron field as

$$P{=}\frac{1}{\rho}\left(\begin{array}{cc}\Psi_{\downarrow}^{\dagger}\Psi_{\downarrow} & \alpha\\ \alpha^{\dagger} & \Psi_{\uparrow}^{\dagger}\Psi_{\uparrow}\end{array}\right),\ P^{2}=P, \tag{27}$$

where α can be an arbitrary function of Ψ_{σ} and $\Psi_{\sigma}^{\dagger}$ satisfying $P^{2}=P$. Arbitrariness of α will not play any role in the following discussions. Topological charge for the case $\langle\mathrm{tr}P\rangle_{\infty}=1$ is

$$Q=\frac{1}{2\pi\theta}\mathrm{Tr}_{\mathcal{H}}\left(\mathrm{tr}P-\langle\mathrm{tr}P\rangle_{\infty}\right)=\frac{1}{2\pi\theta}\mathrm{Tr}_{\mathcal{H}}\left(\mathrm{tr}P-1\right), \tag{28}$$

which can be expressed in terms of the total number of magnetic fluxes N_{ϕ} and the number of electrons N_{e} as

$$Q=\frac{1}{2\pi\theta}\mathrm{Tr}_{\mathcal{H}}\left(\mathrm{tr}P-1\right)=N_{e}-N_{\phi}. \tag{29}$$

Here use has been made of

$$\mathrm{tr}P=\frac{1}{\rho}\left(\Psi_{\downarrow}^{\dagger}\Psi_{\downarrow}+\Psi_{\uparrow}^{\dagger}\Psi_{\uparrow}\right) \tag{30}$$

and

$$N_{e}=\mathrm{Tr}_{\mathcal{H}}\left(\Psi_{\downarrow}^{\dagger}\Psi_{\downarrow}+\Psi_{\uparrow}^{\dagger}\Psi_{\uparrow}\right). \tag{31}$$

On the other hand, as $(2\pi\theta)^{-1}$ is the number of states per unit area of LLL, which is occupied by a unit flux, N_{ϕ} can be considered to be the total number of states in the LLL. And the filling factor is defined as the number of electrons per unit flux, $\nu=\frac{N_{e}}{N_{\phi}}$. Consequently, the topological number has a simple meaning of the electron number added to (removed from) the state with filling factor $\nu=1$.

We consider the delta function like Coulomb repulsion potential between the electrons, $V\left(|z-z'|\right)=\rho^{-1}\delta^{2}\left(z-z'\right)$. Restricting the electron states

to the LLL, the Hamiltonian can be written as

$$
\begin{aligned}
H &= \frac{1}{\rho}\int \left(\Psi_\uparrow^\dagger \Psi_\uparrow - \Psi_\downarrow \Psi_\downarrow^\dagger\right)^2 d^2x \\
&= \frac{2}{\rho}\int \left(\Psi_\uparrow \Psi_\downarrow\right)^\dagger \left(\Psi_\uparrow \Psi_\downarrow\right) d^2x + (-Q) \\
&= \rho \int \left(\mathrm{tr}P - 1\right)^2 d^2x,
\end{aligned} \tag{32}
$$

which is invariant under particle hole exchange [12]. Then the energy is

$$
E = \rho \mathrm{Tr}_{\mathcal{H}} \left(\mathrm{tr}P - 1\right)^2. \tag{33}
$$

We can compare our argument with that of ref.[12] which goes as follows. The Hamiltonian for the above system written in terms of $*$product is expanded in θ and this leads to the $O(3)\sigma$ model on the commutative space. In this paper we shall discuss the problem of solitons working directly within the operator formalism.

The energy inequality for $Q > 0$ can be written as

$$
\begin{aligned}
E &= \rho \mathrm{Tr}_{\mathcal{H}} (\mathrm{tr}P - 1)^2 \\
&= \rho \mathrm{Tr}_{\mathcal{H}} \left\{ (\mathrm{tr}P - 1)^2 - (\mathrm{tr}P - 1) \right\} + \rho \mathrm{Tr}_{\mathcal{H}} (\mathrm{tr}P - 1) \\
&= \rho \mathrm{Tr}_{\mathcal{H}} \left\{ (\mathrm{tr}P - 1)^2 - (\mathrm{tr}P - 1) \right\} + Q \\
&\geq Q,
\end{aligned} \tag{34}
$$

where the first term on the third line is positive definite, as can be seen from

$$
\rho \mathrm{Tr}_{\mathcal{H}} \left\{ (\mathrm{tr}P - 1)^2 - (\mathrm{tr}P - 1) \right\} = \frac{2}{\rho} \mathrm{Tr}_{\mathcal{H}} \left\{ \left(\Psi_\downarrow \Psi_\uparrow\right) \left(\Psi_\downarrow \Psi_\uparrow\right)^\dagger \right\} \geq 0. \tag{35}
$$

Consequently the BPS equation for $Q > 0$ is

$$
(\mathrm{tr}P - 1)^2 = \mathrm{tr}P - 1. \tag{36}
$$

For $Q < 0$ the energy inequality is

$$
\begin{aligned}
E &= \rho \mathrm{Tr}_{\mathcal{H}} (\mathrm{tr}P - 1)^2 \\
&= \rho \mathrm{Tr}_{\mathcal{H}} \left\{ (\mathrm{tr}P)^2 - (\mathrm{tr}P) \right\} - \rho \mathrm{Tr}_{\mathcal{H}} (\mathrm{tr}P - 1) \\
&= \rho \mathrm{Tr}_{\mathcal{H}} \left\{ (\mathrm{tr}P)^2 - (\mathrm{tr}P) \right\} - Q \\
&\geq -Q
\end{aligned} \tag{37}
$$

and we obtain the BPS equation,

$$
(\mathrm{tr}P)^2 = \mathrm{tr}P. \tag{38}
$$

The systems described by (6) and (33) are different, thus the BPS eqs. (10) (11) and (36) (38) are different as they should be. As we shall see in the following, however, the BPS eq. (10) and (36) (anti-BPS eq. (11) and (38)) have common soliton (anti-soliton) solutions.

An example of the BPS soliton for electron system with $Q = n > 0$ is

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \sum_{m=0}^{n-1} |m\rangle \langle m| \end{pmatrix}, \tag{39}$$

which has the energy $E = n$. An example of BPS anti-soliton with $Q = -n < 0$ is

$$P = \begin{pmatrix} 1 - \sum_{m=0}^{n-1} |m\rangle \langle m| & 0 \\ 0 & 0 \end{pmatrix}, \tag{40}$$

which has the energy $E = n$. These soliton solutions are at the same time the solutions of the nonlinear sigma model [10]. On the other hand, the following soliton solutions of the nonlinear sigma model [4]

$$P = \begin{pmatrix} z^n \dfrac{1}{\bar{z}^n z^n + 1} \bar{z}^n & z^n \dfrac{1}{\bar{z}^n z^n + 1} \\ \dfrac{1}{\bar{z}^n z^n + 1} \bar{z}^n & \dfrac{1}{\bar{z}^n z^n + 1} \end{pmatrix} \tag{41}$$

and

$$P = \begin{pmatrix} \bar{z}^n \dfrac{1}{z^n \bar{z}^n + 1} z^n & \bar{z}^n \dfrac{1}{z^n \bar{z}^n + 1} \\ \dfrac{1}{z^n \bar{z}^n + 1} z^n & \dfrac{1}{z^n \bar{z}^n + 1} \end{pmatrix} \tag{42}$$

do not satisfy the BPS equations (36) or (38), and thus are not the solitons of the electron system, although these configurations do have the finite energy.

Next, we comment on more general form of the soliton solutions in the electron system. Non-BPS anti-soliton of ref.[9]

$$P = \begin{pmatrix} 1 - |m\rangle \langle m| & 0 \\ 0 & 0 \end{pmatrix} \tag{43}$$

is also a BPS soliton solution with $Q = -1$ and $E = 1$. As far as $\mathrm{tr}P$ is unchanged, arbitrary deformations of (39), (40) and (43) are also the soliton solutions.

4 Classification by New Topological Number

In section 2, we have seen that the solitons of nonlinear σ model on the noncommutative plane are classified by the topological charge Q and the new topological number $\langle \mathrm{tr}P \rangle_\infty$. Based upon this situation, in section 3, we have found that some solitons of nonlinear σ model with $\langle \mathrm{tr}P \rangle_\infty = 1$

can be considered as solitons in the model of electron system with a filling factor $\nu \simeq 1$.

What are the solitons with $\langle \mathrm{tr}P \rangle_\infty = 0, 2$? As they have the topological charges

$$Q = \frac{1}{2\pi\theta} \mathrm{Tr}_{\mathcal{H}} \left(\mathrm{tr}P \right) = N_e \quad \text{for} \quad \langle \mathrm{tr}P \rangle_\infty = 0 \tag{44}$$

and

$$Q = \frac{1}{2\pi\theta} \mathrm{Tr}_{\mathcal{H}} \left(\mathrm{tr}P - 2 \right) = N_e - 2N_\phi \quad \text{for} \quad \langle \mathrm{tr}P \rangle_\infty = 2, \tag{45}$$

respectively, the different topological numbers $\langle \mathrm{tr}P \rangle_\infty$ imply that these solitons belong to different vacua with filling factors $\nu \gtrsim 0$, $\nu \lesssim 2$.

Hamiltonians for the electron system corresponding to the solitons with $\langle \mathrm{tr}P \rangle_\infty = 0, 2$ can be written as

$$\begin{aligned} H_0 &= \frac{1}{\rho} \int \left(\Psi_\uparrow^\dagger \Psi_\uparrow + \Psi_\downarrow^\dagger \Psi_\downarrow \right)^2 d^2x \\ &= \rho \int \left(\mathrm{tr}P \right)^2 d^2x \end{aligned} \tag{46}$$

and

$$\begin{aligned} H_2 &= \frac{1}{\rho} \int \left(\Psi_\uparrow \Psi_\uparrow^\dagger + \Psi_\downarrow \Psi_\downarrow^\dagger \right)^2 d^2x \\ &= \rho \int \left(\mathrm{tr}P - 2 \right)^2 d^2x, \end{aligned} \tag{47}$$

respectively. We note, however, that the Coulomb repulsion could be negligibly small for $\nu \approx 0$, but for the sake of simplicity of our argument we shall assume its existence for the whole region of $0 \lesssim \nu \lesssim 2$. Then the energy for $\langle \mathrm{tr}P \rangle_\infty = 0$ can be expressed as

$$\begin{aligned} E_0 &= \rho \mathrm{Tr}_{\mathcal{H}} (\mathrm{tr}P)^2 \\ &= \rho \mathrm{Tr}_{\mathcal{H}} \left\{ (\mathrm{tr}P)^2 - (\mathrm{tr}P) \right\} + \rho \mathrm{Tr}_{\mathcal{H}} (\mathrm{tr}P) \\ &= \rho \mathrm{Tr}_{\mathcal{H}} \left\{ (\mathrm{tr}P)^2 - (\mathrm{tr}P) \right\} + Q \\ &\geq Q, \end{aligned} \tag{48}$$

which leads to the BPS equation

$$(\mathrm{tr}P)^2 = \mathrm{tr}P. \tag{49}$$

In this case, we have solitons with $Q > 0$, the concrete example of which with $Q = n$ and $E = n$ is expressed by the projector

$$P = \begin{pmatrix} \sum_{m=0}^{n-1} |m\rangle \langle m| & 0 \\ 0 & 0 \end{pmatrix}. \tag{50}$$

Similarly, the energy for $\langle \mathrm{tr}P \rangle_\infty = 2$ is

$$
\begin{aligned}
E_2 &= \rho \mathrm{Tr}_{\mathcal{H}} (\mathrm{tr}P - 2)^2 \\
&= \rho \mathrm{Tr}_{\mathcal{H}} \left\{ (\mathrm{tr}P - 2)^2 + (\mathrm{tr}P - 2) \right\} - \rho \mathrm{Tr}_{\mathcal{H}} (\mathrm{tr}P - 2) \\
&= \rho \mathrm{Tr}_{\mathcal{H}} \left\{ (2 - \mathrm{tr}P)^2 - (2 - \mathrm{tr}P) \right\} - Q \\
&\geq -Q \ , \qquad (51)
\end{aligned}
$$

and the BPS equation is

$$
(2 - \mathrm{tr}P)^2 = 2 - \mathrm{tr}P. \qquad (52)
$$

Then there exist solitons with $Q < 0$, and the example of which with $Q = -n$ and $E = n$ is given by

$$
P = \begin{pmatrix} 1 - \sum_{m=0}^{n-1} |m\rangle \langle m| & 0 \\ 0 & 1 \end{pmatrix}. \qquad (53)
$$

These soliton solutions (50) and (53) are at the same time the solutions of the nonlinear sigma model [11].

Thus, we have found that, as in the case of $\langle \mathrm{tr}P \rangle_\infty = 1$, some solitons of the nonlinear sigma model with $\langle \mathrm{tr}P \rangle_\infty = 0, 2$ can also be considered to be the solitons of the electron system.

5 Summary and Discussion

In this paper, we have seen that some solitons of nonlinear σ model on the noncommutative plane can be considered as solitons in the model of two dimensional electron system. This is rather surprising, because the BPS equations for the electron system on one hand and the nonlinear σ model on the other are very different. These solitons are classified by two topological numbers Q and $\langle \mathrm{tr}P \rangle_\infty = 0, 1, 2$, as is described in section 2. Corresponding to these topological numbers, in section 3 and 4, we have obtained the Hamiltonians, BPS equations and soliton solutions of the electron system. The noncommutative solitons with $\langle \mathrm{tr}P \rangle_\infty = 0, 1, 2$ correspond to the solitons with a filling factor $\nu \gtrsim 0, \nu \simeq 1$ and $\nu \lesssim 2$, respectively.

It would be interesting to find the corresponding experimental evidence for the soliton solutions of the electron system.

It is instructive to compare our noncommutative soliton solutions with those of refs.[12][13], where they have arrived at the $O(3)\sigma$ model on commutative space by expanding the Hamiltonian, written in terms of the $*$product, in θ. Thus their solitons are those of commutative model. In present paper, on the other hand, we have discussed the problem of solitons working directly within the operator formalism. As we have seen, these solitons are different from the commutative limits of our noncommutative solitons.

Finally, in connection with the recent work [17] the stability of our solutions have to be examined.

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